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Controllability of Fractional Neutral Impulsive Integro-Differential Equations in Banach Spaces

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ABSTRACT

In this paper, we prove the controllability for impulsive neutral fractional integrodierential equations in Banach spaces using Banach fixed point theorem, semigroup theory and fractional calculus. We present the controllability result by introducing a class of distributed controls, which are highly useful for the computational purpose also. The controversy on the solution operator is discussed here and we emphasize that we use the generalized Caputo derivative with the lower bound at zero for the system considered. An example is given to illustrate the abstract results.

***Keywords**: Controllability; Impulse; Fractional differential equations; Integro-differential equations; Fixed-point theorem.*

INTRODUCTION

The use of fractional order derivatives and integrals in control theory leads to better results than integer order approaches. Fractional calculus has received great attention, because fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various processes. In particular, fractional differential equations draw great applications in many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic models. In addition, the study of fractional differential equations (FDE) have gained considerable importance due to their application. For more details of fractional differential equations with applications and about fractional calculus, see the monographs of [1, 5, 12, 14] and the papers of [3, 7, 8].

The concept of controllability plays an important role in various areas of science and engineering. It is one of the fundamental concept of mathematical control theory. Controllability of nonlinear systems infinite dimensional spaces has been studied extensively by means of fixed-point techniques. More exactly, the problem of controllability deals with the existence of a control function, which steers the solution of the system from its initial state to a _nal state. Many of the authors has been giving importance to this study and doing research under it, see for instance [13, 18, 22, 25].

The investigation of impulsive functional differential or integro-differential frameworks is interesting for their application in strengthening techniques and phenomena conditional on short-time perturbations in the course of their progress. The perturbations are conducted separately and their term is insignificant in correlation with the aggregate length of time of the procedures. The study of impulsive differential equation is linked to their utility in simulating processes and phenomena subject to short time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. We recall that numerous impulsive frameworks emerging from sensible models can be depicted as partial differential equations with non-instantaneous impulses (NII) in Banach spaces. There are various ways to investigate this sort of models. Kumar et al. [32] analyzed the existence of solutions for FDE with NII in Banach spaces through the utilization of the appropriate fixed-point theorem (FPT). Gautam and Dabas [33] acknowledged the existence of mild solutions for fractional integro differential equation (FIDE) with state-dependent delay (SDD) and NII through the employment of the suitable fixed-point theorem (FPT). Also, they proved researched the existence of mild solution of a class of second order partial neutral differential equations with SDD and NII in Banach spaces. This motivated us to investigate the existence results of the system with NII in Banach spaces. Thus impulsive integro-differential equations have become an important object of investigation in recent years stimulated by their numerous applications to problems in mechanics, electrical engineering, medicine, biology, ecology etc., we refer [11, 19, 21].

Recently, Tai and Wang [23] discussed the controllability of fractional order impulsive neutral functional infinite delay integro-differential systems in Banach spaces using fractional calculus, a semigroup of operators and Krasnoselskii's fixed-point theorem. Ganesh et al. [9] studied the controllability of neutral fractional functional equations with impulses and infinite delay by using solution operator, fractional calculations and fixed-point techniques. Shu et al. [29] established the existence and uniqueness of solutions for class of fractional partial semi linear functional differential equations with finite delay by using Banach fixed-point theorem. H. Qinat el. [34] studied the controllability result of the fraction order system using measure of noncompactness and H. Qin at el. ([35], [36]) studied the approximate controllability and optimal controls of fractional evolution and dynamical systems in abstract spaces. Ravichandran and Trujillo [20] investigated the controllability of impulsive fractional functional integro-differential equations in Banach spaces using solution operator and fixed point theorems. Very recently, Liu and Li [13] established the controllability of impulsive fractional evolutions inclusions in Banach spaces by using the well-known fixed-point theorem for multivalued maps due to Dhage associated with an evolution system. Here authors studied the controllability result of impulsive fractional evolution inclusion without delay argument. Also, the example given is not in the explicit form. In our opinion, the example should have been given with full justification of the assumptions. We have given more general example to fill up this gap in section 4. Also[18] proved the existence and controllability results for fractional impulsive integro-differential systems in Banach spaces using fixed point theorem, semigroup theory ad generalized Bellman inequality, but the system considered is not neutral and without delay term. Also this paper generalized the recent work of S. Liang and R. Mei (refer [30]).

Motivated by the above mentioned works [20, 23, 30], in this paper, we prove the control-lability results for fractional impulsive neutral integro-differential systems with infinite delay in Banach spaces of the form

$$\begin{cases} D_t^q[x(t)-g(t,x_t)] = Ax(t) + f\left(t, x_t, \int_0^t h(t,s,x_s)ds\right) + Bu(t), \ t\in I=[0,a], \ t\neq t_k \\ x(0)=x_0 \text{ on } [-r,0], \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k=1,2,\cdots,m; \end{cases} \tag{1.1}$$

Where D_t^q is the Caputo derivative, 0 < q < 1. Here A: D(A) $\subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}, t \geq 0$ of a uniformly bounded operator on X, and A is a bounded linear operator. The history, x_t : $[-r,0] \to X$ are defined by $x_t(\theta) = x(t=\theta)$ belongs to a Banach Space X . f : $I \times X \times X \to X$ is given X value function, $h: I \times I \times X \to X$ are continuous, here $I_k : X \to X$ are impulsive function,

$$0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a, \ \Delta_x|_{t=t_k} = x(t_k^+),$$
$$- x(t_k^-), \ x(t_k^+) = \lim_{h\to 0+} x(t_k + h)$$

and $x(t_k^-) = \lim_{h\to 0-} x(t_k + h)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively and B is a bounded linear operator from U to X, and the control function $u \in L^2[I, X]$ with U as a Banach space.

The purpose of this paper is to study the controllability of fractional neutral impulsive integro-differential equations in Banach spaces with nite delay by introducing a class of control instead of just one control, using the characteristic solution operator and without using the compactness conditions on the semigroup $\{T(t)\}, \ t \geq 0$. The advantages of introducing class of control is to compute the solution numerically also, that is to compute a class of control as much as the real life control problems like space aircraft, missile problems are concerned. To best of our knowledge, this is an untreated article as far as distributed control is concerned for fractional neutral integro-differential systems with time delay argument.

The current paper has the following subsections. In section 2, we present some preliminary lemmas and defnitions, which will be used to prove our main results. In section 3, we present the controllability results for fractional impulsive neutral integro-differential systems (1.1) through by introducing a class of controls using Banach FPT and semigroup theory and an example is given in section 4 to illustrate the application of the abstracts.

PRELIMINARIES

Throughout this paper, let A be the infinitesimal generator of a C_0 semigroup $\{T(t)\}, \ t \geq 0$ of uniformly bounded operators on X. Let us consider the set of functions.

$$\mathbb{PC}[I,\mathbb{X}] = \Big\{x : I \to \mathbb{X} \mid x \in C[(t_k, t_{k+1}), \mathbb{X}] \text{ and there exist } x(t_k^+) \text{ and } x(t_k^-), \\ \text{with } x(t_k^-) = x(t_k); k = 0,1,2,\cdots,m\Big\}.$$

Endowed with the norm $\|x\|_{PC} = \sup_{t \in I} \|x(t)\|$, it is easy to know that $(PC[I, X], \|\cdot\|_{PC})$ is a Banach space. Let $L_B(X)$ be the Banach space of all linear and bounded operator on X. For a C_0-semigroup $\{T(t)\},\ t \geq 0$, we set $M_1 = \sup_{t \in I} \|T(t)\|_{L_B(X)}$. For each positive constant r, set $B_r = \{x \in PC[I, X] : \|x\| \leq r\}$.

Definition 2.1. *[10]* The fractional integral of γ order with the lower limit zero for a function f is defined as

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0,\ \gamma > 0.$$

Provided that the right side is point-wise defined on $[0, +\infty)$, where $\Gamma(.)$ is the gamma function.

Definition 2.2. [10] The Riemann-Lioville derivative of the order γ with the lower limit zero for a function $f : [0, \infty) \to R$ can be written as

$${}^L D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{1-n+\gamma}} ds,\ t > 0,\ n-1 < \gamma < n.$$

Definition 2.3. [10] The Caputo derivative of the order γ for a function $f : [0, \infty) \to R$ can be written as

$$D^\gamma f(t) = {}^L D^\gamma \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), t > 0,\ n-1 < \gamma < n.$$

Remark 2.1. *[10]*

(1) If $f(t) \in C^n[0, \infty)$, then

$$\begin{aligned} D^\gamma f(t) &= \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1-n+\gamma}} ds \\ &= I^{n-r} f^{(n)}(t),\ t > 0,\ n-1 < \gamma < n. \end{aligned}$$

(2) The Caputo derivative of a constant is equal to zero.

(3) If f is an abstract function with values in X, then integrals, which appear in definitions 2.1, 2.2 and 2.3 are taken in Bochner's sense.

Definition 2.4. [27]. A mild solution of the following non-homogeneous impulsive linear fractional equation of the form

$$\begin{cases} D_t^q x(t) &= Ax(t) + h(t),\quad t \in I = [0, b],\ 0 < q < 1,\ t \neq t_k, \\ x(0) &= x_0 \in \mathbb{X}, \\ \Delta x|_{t=t_k} &= I_k(x(t_k^-)),\ k = 1, 2, \cdots, m, \end{cases} \tag{2.1}$$

Is given by

$$x(t) = \begin{cases} \mathcal{U}(t)x_0 + \int_0^t (t-s)^{q-1} \mathcal{V}(t-s) h(s) ds,\ t \in [0, t_1], \\ \mathcal{U}(t)x_0 + \mathcal{U}(t-t_1) I_1(x(t_1^-)) + \int_0^t (t-s)^{q-1} \mathcal{V}(t-s) h(s) ds,\ t \in (t_1, t_2], \\ \vdots \\ \mathcal{U}(t)x_0 + \sum_{k=1}^m \mathcal{U}(t-t_k) I_k(x(t_k^-)) + \int_0^t (t-s)^{q-1} \mathcal{V}(t-s) h(s) ds,\ t \in (t_m, b], \end{cases} \tag{2.2}$$

Where $U(\bullet)$ and $\nu(\bullet)$ are called characteristic solution operators and given by

$$\mathcal{U}(t) = \int_0^\infty \xi_q(\theta) T(t^q \theta) d\theta,\ \textit{and}\ \mathcal{V}(t) = q \int_0^\infty \theta \xi_q(\theta) T(t^q \theta) d\theta, \tag{2.3}$$

And for $\theta \in (0, \infty)$,

$$\xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \varpi_q(\theta^{-\frac{1}{q}}) \geq 0,\ \varpi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} sin(n\pi q),$$

Remark 2.2. Controversy on the solution operator, Definition 2.5, based on Definition 2.3 and Definition 2.4

1. In this paper we emphasize that we use the generalized Caputo derivative with the lower bound at zero for the equation (1.1). However, we have not chosen the classical Caputo derivative and have not changed it in each sub-intervals for the equation (1.1), where the impulses start at the lower bound t_k: Obviously, we mean keeping a different one, in each of the impulses the lower bound is at zero. Moreover, Definition 2.5 is more reasonable since the generalized Caputo derivative in the equation (1.1) should be fixed at the lower bound at zero once we set initial time at zero. Therefore, we do not expect to change the lower bound repeatedly in the definition of Caputo derivative for the same equation.

2. We use Definition 2.4 (generalized Caputo derivative), where the integrable function f can be discontinuous. Definition 2.4 is more general with respect to Remark 2.1 (i) (relationship between strong and weak Caputo derivatives). Therefore, result would be wrong if we have used strong Caputo derivative.

Definition 2.5. By a $\mathbb{PC}$ mild solution of the problem (1.1), we mean that a function x 2 PC [I;$\mathbb{X}$] satisfies the following integral equation:

$$x(t)=\begin{cases}\mathcal{U}(t)[x_0-g(0,x_0)]+g(t,x_t)+\int_0^t(t-s)^{q-1}A\mathcal{V}(t-s)g(s,x_s)ds\\ \quad+\int_0^t(t-s)^{q-1}\mathcal{V}(t-s)\left[f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)+Bu(s)\right]ds,\\ \qquad\qquad t\in[0,t_1],\\ \mathcal{U}(t)[x_0-g(0,x_0)]+\mathcal{U}(t-t_1)I_1(x(t_1^-))\\ +g(t,x_t)+\int_0^t(t-s)^{q-1}A\mathcal{V}(t-s)g(s,x_s)ds\\ +\int_0^t(t-s)^{q-1}\mathcal{V}(t-s)\left[f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)+Bu(s)\right]ds,\\ \qquad\qquad t\in(t_1,t_2],\\ \vdots\\ \mathcal{U}(t)[x_0-g(0,x_0)]+\sum_{i=1}^m\mathcal{U}(t-t_i)I_i(x(t_i^-))+g(t,x_t)\\ +\int_0^t(t-s)^{q-1}A\mathcal{V}(t-s)g(s,x_s)ds\\ +\int_0^t(t-s)^{q-1}\mathcal{V}(t-s)\left[f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)+Bu(s)\right]ds,\\ \qquad\qquad t\in(t_m,a].\end{cases}$$

Definition 2.6. [17] Let X be a Banach space, a one-parameter family $\{T(t)\}, t\geq 0$ of bounded linear operators from X to X is a semigroup of bounded linear operators on $\mathbb{X}$ if

(1) T(0) = I, (here I is the identity operator on $\mathbb{X}$)

(2) T(t + s) = T(t)T(s) for every $t, s \geq 0$ (the semigroup property)

(3) A semigroup of bounded linear operator, {T(t)}, is uniformly continuous if

$$\lim_{t\downarrow 0}||T(t)-I||=0.$$

Lemma 2.1. [17] Linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is the bounded linear operator.

Lemma 2.2. [28] The operators U(t) and $\mathcal{V}(t)$ have the following properties:

(i) For any fixed $t\geq 0, U(t)$ and $\mathcal{V}(t)$ bounded linear operators, i.e. for any $x\in\mathbb{X}$,

$$||\mathcal{U}(t)x||\leq M_1||x||,\quad ||\mathcal{V}(t)x||\leq\frac{qM_1}{\Gamma(1+q)}||x||.$$

(ii) $\{u(t), t\geq 0\}$ and $(\mathcal{V}(t), t\geq 0\}$ are uniformly continuous, that is, for each fixed t > 0, and $\in\ > 0$ there exists h > 0 such that

$$||\mathcal{U}(t+\epsilon)-\mathcal{U}(t)||\leq\epsilon,\ \ for\ t+\epsilon\geq 0\ and\ |\epsilon|<h,$$
$$||\mathcal{V}(t+\epsilon)-\mathcal{V}(t)||\leq\epsilon,\ \ for\ t+\epsilon\geq 0\ and\ |\epsilon|<h.$$

CONTROLLABILITY RESULTS

In this section, we give the existence of controllability results of the system (1.1). To establish our results, we introduce the following hypotheses:

(H1) $f: I\times\mathbb{X}\times\mathbb{X}\to\mathbb{X}$ is continuous, and there exists functions $\alpha_1,\ \alpha_2\ \in L[I,R^+]$ such that

$$||f(t,x_1,x_2)-f(t,y_1,y_2)||\leq\alpha_1(t)||x_1-y_1||+\alpha_2(t)||x_2-y_2||,\ \forall x_i,y_i\in\mathbb{X},\ i=1,2.$$

(H2) $h: I\times I\times\mathbb{X}\to\mathbb{X}$ is continuous and there exist $L_h > 0$ such that

$$||h(t,s,x_1)-h(t,s,y_1)||\leq L_h||x_1-y_1||_{\mathbb{PC}},\ \forall x_i,\ y_i\in\mathbb{X},\ i=1,2.$$

(H3) $g: I\times\mathbb{X}\to\mathbb{X}$ and there exist positive constants G1; G2 > 0 such that

$$||g(t,x_1)-g(t,y_1)||\leq G_1\ ||x_1-y_1||_{\mathbb{PC}},$$
$$||Ag(t,x_1)-Ag(t,y_1)||\leq G_2\ ||x_1-y_1||_{\mathbb{PC}},\forall x_1,y_1\in PC([0,a],\mathbb{X}).$$

(H4) The function $I_k:\ \mathbb{X}\to\mathbb{X}$ are continuous and there exist $\mu_k > 0$ such that

$$||I_k(x)-I_k(y)||\leq\mu_k||x-y||,\quad x,y\in\mathbb{X},\ k=1,2,\cdots,m.$$

(H5) The function $\Delta_m(t):\ I\to\mathbb{R}^+$ is defined by

$$\Delta_m(t)=N\left[G_1+\frac{M_1a^q}{\Gamma(1+q)}(G_2+\alpha_1(t)+\alpha_2(t)L_ha)+M_1\mu_k\right];\ 0<\Delta_m(t)<1,\ t\in I.$$

$$\Delta'_m(t)=\left[G_1+\frac{M_1a^q}{\Gamma(1+q)}(G_2+\alpha_1(t)+\alpha_2(t)L_ha+\Delta_m(t))+M_1m\mu_m\right];\ 0<\Delta'_m(t)<1.$$

Definition 3.7.

The system (1.1) is said to be controllable on the interval *I*, if for every initial condition x_0 and $x_1\in\mathbb{X}$, there exists a control $u\in L^2(I,U)$ such that the solution $x(\bullet)$ of (1.1) satisfies $x(a)=x_1$.

We present the controllability results for fractional neutral impulsive integro-differential systems (1.1), by introducing a class of controls.

(H6) The linear operator W_i from $L^2[(0,t_i],U]$ into $\mathbb{X}$ defined by

$$W_i u = \int_0^{t_i} (t_i - s)^{q-1}\mathcal{V}(t_i - s)Bu(s)ds,\ i = 1, 2, \cdots m, m+1;$$

Induces an invertible operator W_i^{-1} defined on $L^2[(0,t_i],U]/\ker W_i$ and there exists a positive constant N > 0 such that $\| BW_i^{-1}\| \le N$.

Theorem 3.1. If the hypotheses (H1)-(H5) and (H6) are satisfied, then the fractional neutral impulsive integro-differential system (1.1) is controllable on I.

Proof: Using the hypothesis (H6), for an arbitrary function $x(\bullet)$, define the control

$$u(t) = \begin{cases} W_1^{-1}\left[x_0 + \frac{x_1 - x_0}{m+1} - \mathcal{U}(t_1)[x_0 - g(0,x_0)]\ - g(t,x_t)\right. \\ -\int_0^{t_1}(t_1 - s)^{q-1}A\mathcal{V}(t_1 - s)g(s,x_s)ds \\ \left.-\int_0^{t_1}(t-s)^{q-1}\mathcal{V}(t-s)\left[f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)\right]ds\right](t),\ t\in[0,t_1], \\ W_2^{-1}\left[x_0 + 2\frac{x_1 - x_0}{m+1} - \mathcal{U}(t_2)[x_0 - g(0,x_0)] - \mathcal{U}(t_2 - t_1)I_1(x(t_1^-))\right. \\ -g(t,x_t). - \int_0^{t_2}(t_2 - s)^{q-1}A\mathcal{V}(t_2 - s)g(s,x_s)ds \\ \left.-\int_0^{t_2}(t_2 - s)^{q-1}\mathcal{V}(t_2 - s)\left[f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)\right]ds\right](t),\ t\in(t_1,t_2], \\ \vdots \\ W_{m+1}^{-1}\left[x_1 - \mathcal{U}(a)[x_0 - g(0,x_0)] - \sum_{i=1}^{m}\mathcal{U}(a - t_i)I_i(x(t_i^-)) - g(t,x_t)\right. \\ -\int_0^{a}(a-s)^{q-1}A\mathcal{V}(a-s)g(s,x_s)ds \\ \left.-\int_0^{a}(a-s)^{q-1}\mathcal{V}(a-s)\left[f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)\right]ds\right](t),\ t\in(t_m,a]. \end{cases} \quad (3.1)$$

Define the operator $F : \mathbb{PC}[I,\mathbb{X}] \to \mathbb{PC}[I,\mathbb{X}]$,

$$(Fx)(t) = \begin{cases} \mathcal{U}(t)[x_0 - g(0,x_0)] + g(t,x_t) + \int_0^t (t-s)^{q-1}A\mathcal{V}(t-s)g(s,x_s)ds \\ \quad + \int_0^t (t-s)^{q-1}\mathcal{V}(t-s)\left[f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau) + Bu(s)\right]ds,\ t\in[0,t_1], \\ \mathcal{U}(t)[x_0 - g(0,x_0)] + \mathcal{U}(t-t_1)I_1(x(t_1^-)) + g(t,x_t) \\ \quad + \int_0^t (t-s)^{q-1}A\mathcal{V}(t-s)g(s,x_s)ds \\ + \int_0^t (t-s)^{q-1}\mathcal{V}(t-s)\left[f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau) + Bu(s)\right]ds,\ \ t\in(t_1,t_2], \\ \vdots \\ \mathcal{U}(t)[x_0 - g(0,x_0)] + \sum_{i=1}^{m}\mathcal{U}(t-t_i)I_i(x(t_i^-)) + g(t,x_t) \\ \quad + \int_0^t (t-s)^{q-1}A\mathcal{V}(t-s)g(s,x_s)ds \\ + \int_0^t (t-s)^{q-1}\mathcal{V}(t-s)\left[f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau) + Bu(s)\right]ds,\ \ t\in(t_m,a]. \end{cases}$$

We shall prove that the operator F has a fixed point, when using the above control. Clearly this fixed point is a PC-mild solution of the control system (1.1) and $x(a) = x_1$, that is the control we defined steers the system (1.1) from initial x0 to x1 in time a.

For any $x_1, x_2 \in C[(t_i, t_{i+1}], \mathbb{X}\ (i = 0, 1, 2\cdots, m)$, ssby conditions (H1)-(H6), we get

$$\begin{aligned} \|Bu_1(t) - Bu_2(t)\| \le\ & \|BW^{-1}\|\ [\|g(t,x_{1_t}) - g(t,x_{2_t})\| \\ & + \int_0^t (a-s)^{q-1}\|\mathcal{V}(t-s)\|\ \|Ag((s,x_{1_s}) - Ag(s,x_{2_s})\|\ ds \\ & + \int_0^t (a-s)^{q-1}\|\mathcal{V}(t-s)\| \\ & \times \|f(s,x_{1_s},\int_0^s h(s,\tau,x_{1_\tau})d\tau) - f(s,x_{2_s},\int_0^s h(s,\tau,x_{2_\tau})d\tau)\|ds \\ & + \|\mathcal{U}(t-t_i)\|\ \|I_i(x_1(t_i^-)) - I_i(x_2(t_i^-))\|] \\ \le\ & N\left[G_1 + \frac{M_1 G_2 a^q}{\Gamma(1+q)} + \frac{M_1 a^q}{\Gamma(1+q)}(\alpha_1(t) + \alpha_2(t)L_h a) + M_1\mu_k\right] \\ & \|x_1(s) - x_2(s)\|_{\mathbb{PC}} \\ \le\ & \Delta_m(t)\|x_1(s) - x_2(s)\|_{\mathbb{PC}} \end{aligned}$$

and

$$\begin{aligned} \|(Fx_1)(t) - (Fx_2)(t)\| \le\ & \|g(t,x_{1_t}) - g(t,x_{2_t})\| \\ & + \int_0^t (a-s)^{q-1}\|\mathcal{V}(t-s)\|\ \|Ag((s,x_{1_s}) - Ag(s,x_{2_s})\|\ ds \\ & + \int_0^t (a-s)^{q-1}\|\mathcal{V}(t-s)\|\ \|f(s,x_{1_s},\int_0^s h(s,\tau,x_{1_\tau})d\tau) \\ & \quad - f(s,x_{2_s},\int_0^s h(s,\tau,x_{2_\tau})d\tau)\|ds \\ & + \int_0^t (t-s)^{q-1}\|\mathcal{V}(t-s)\|\ \|Bu_1(s) - Bu_2(s)\|ds \\ & + \|\mathcal{U}(t-t_i)\|\ \|I_i(x_1(t_i^-)) - I_i(x_2(t_i^-))\| \\ \le\ & \left[G_1 + \frac{M_1 G_2 a^q}{\Gamma(1+q)} + \frac{M_1 a^q}{\Gamma(1+q)}(\alpha_1(s) + \alpha_2(s)L_h a)\right. \\ & \left.\frac{M_1 a^q}{\Gamma(1+q)}\Delta_m(t) + M_1 m\mu_m\right]\|x_1(s) - x_2(s)\|_{\mathbb{PC}} \\ \le\ & \Delta'_m(t)\|x_1(s) - x_2(s)\|_{\mathbb{PC}} \end{aligned}$$

Since $0 < \Delta_m^r(t) < 1,$ then F is contraction mapping. Any fixed point of F is a PC-mild solution of (1.1) which satisfies $x(a) = x_1$. Hence, system (1.1) is controllable on I.

Remark: In this paper, we have not used the phase space considered by Hall and Kato [31] because system contains only definite delay. One can extend the same work for the system containing definite delay using the phase space given by Hall and Kato [31] or a new definition of the phase space given in [7].

AN EXAMPLE

Let $\mathbb{X} = (L^2([0,\Pi],\mathbb{R}) \|\ \|_2).$ Consider the following fractional neutral functional integro-differential equations with impulsive conditions of the form:

$$\begin{aligned}
{}^cD_t^q\left[w(t,\eta)-\int_0^\pi d(\eta-r,y)w(t,\eta)dy\right] &= \frac{\partial^2}{\partial t^2}w(t,\eta)+u(t,y)\\
&+ F\left(t,w(t,\eta-r),\int_0^t h(t,v(x,\eta-r))ds\right),\\
&\quad (t,\eta-r)\in[0,T]\times(0,\pi),\ t\neq\frac{T}{2},\\
w(t,0) &= w(t,\pi)=0,\quad t\in[0,T],\\
w(0,\eta) &= w_0(\eta),\quad 0<\eta<\pi,\\
\Delta w|_{t=\frac{T}{2}}=I_1\left(\frac{T^-}{2}\right) &= I\left(x,\int_0^1 w(t_k,y)dy\right), k=1,2,\cdots,m;
\end{aligned}$$

Where $T>0, 0<q<1, {}^cD_t^q$ is a Caputo fractional partial derivative of order $q\in(0,1), 0<\tau_1<\tau_2<\tau_3\cdots<\tau_n<a$ and $0=\tau_1<\tau_2<\tau_3<\cdots<\tau_n=a.$ we define an operator A by $A_v=-v''$ with domain

$D(A)=\{v(\cdot)\in X: v, v'$ are absolutely continuous; $v, v''\in X, v(0)=v(\pi)=0\}.$

Then A generates an analytic semigroup $\{T(t)\}, t\geq 0$ and $\| T(t)\|\leq e^{-t}\leq 1.$ It is well known that $0\in\rho(A),$ and so the fractional powers of A are well defined. Moreover, A has a distinct spectrum with eigenvalues of the form $n^2\pi^2; n\in N$ and the corresponding normalized eigenfunctions are $e_n(x)=\sqrt{2}sin(n\pi x), n=1,2,\cdots;0\leq x\leq\pi.$ In addition $\{e_n: n\in N\}$ is an orthogonal basis for X.

$$T(t)v=\sum_{n=1}^{\infty}e^{-n^2\pi t}<v,e_n>e_n;\ v\in X.$$

Hence $\{T(t)\}, t\leq 0$ is uniformly bounded semigroup.

We define $A^{\frac{1}{2}}$

$$A^{\frac{1}{2}}z=\sum_{n=1}^{\infty}n<z,e_n>e_n$$

For each $z\in D(A^{\frac{1}{2}}):=z(\cdot)\in X:\sum_{n=1}^{\infty}n<z,e_n>e_n.$ from [13] we know that if $z\in D(A^{\frac{1}{2}}),$ then z is absolutely continuous with $z'\in X.$ We define the Banach space $X_{\frac{1}{2}}$ by $X_{\frac{1}{2}}=(D(A^{\frac{1}{2}}),$ where $\| z\|_{\frac{1}{2}}=\| A^2 z\|_2=\| z'\|_2$ for any $z\in D(A^{\frac{1}{2}}).$ It is well known that $\| A^{\frac{1}{2}}\| =1$ is bounded.

For solving the problem (1.1), we need the following assumptions:

(P1) The function $d:[0;1]\times[0;1]\rightarrow R$ satisfies the following conditions:

(a) $(x,y)\rightarrow\frac{\partial^2}{\partial x^2}d(x,y)$ is well defined and measurable with

$$\int_0^1\int_0^1\left(\frac{\partial^2}{\partial x^2}d(x,y)\right)^2dydx<+\infty.$$

(b)

$$d(0,y)=d(1,y)=0,\forall y\in[0,1].$$

(P2) The function $I:[0,1]\times R\rightarrow R$ satisfies the following conditions:

(a) For each $\psi\in R,$ the functional $I(.,\psi)$ is differentiable and $\frac{\partial}{\partial x}I(x,\psi)\in X.$

(b) There exists a constant such that $N_1>0$ such that

$$\left|\frac{\partial}{\partial x}I(x,\psi_1)-\frac{\partial}{\partial x}I(x,\psi_2)\right|\leq N_1|\psi_1-\psi_2|$$

for any $x\in[0,1]$ and $\psi_1,\psi_2\in R$

(c)

$$I(0,y)=I(1,y)=0,\forall y\in R.$$

Let $PC([0,T],X_{\frac{1}{2}})$ be the Banach space equipped with sup norms

$$\|w\|_{PC}=sup_{0\leq t\leq T}\|w(t)(\cdot)\|_{\frac{1}{2}}=sup_{0\leq t\leq T}\|(w(t))'(\cdot)\|_2,$$

Also let

1. $f:[0,T]\times X_{\frac{1}{2}}\times X_{\frac{1}{2}}\to X$ be defined by $f(t,\phi_1,\phi_2)(\cdot)=F(t,\cdot,\phi_1(\cdot),\phi_2(\cdot))$;
2. $g:[0,T]\times X_{\frac{1}{2}}\to X$ be defined by $g(t,\phi)(\cdot)=\int_0^\pi d(\cdot,y)\phi(y-r)dy$;
3. $h:[0,T]^2\times X_{\frac{1}{2}}$ is defined by $h(t,s,\phi)(\cdot)=h(t,v(x,\eta-\gamma))$.
4. $I_k:X_{\frac{1}{2}}\to X$ be defined by $I_k(\phi)(\cdot)=I\left(\cdot,\int_0^1\phi(y)dy\right)$.
5. $w:[0,T]\times[0,\pi]\to R$, be defined by $w(t)(\cdot)=w(t,.)$.

Here $f:[0,T]\times X_{\frac{1}{2}}\times X_{\frac{1}{2}}, g:[0,T]\times X_{\frac{1}{2}}, h:[0,T]^2\times X_{\frac{1}{2}}$ are continuous.

By the definition of g and assumption (P1) a similar computation as in [18] shows that $h\in D(A)$

and

$$\|Ag(t_1,\phi_1)-Ag(t_2,\phi_2)\|_2^2\leq\int_0^1\int_0^1\left(\frac{\partial^2}{\partial x^2}b(x,y)\right)^2dydx.\|\phi_1-\phi_2\|_{\frac{1}{2}}^2$$

for each $(t,\phi_1),(t,\phi_2),\in[0,T]\times X_{\frac{1}{2}}$. Hence g satisfies the hypothesis (H3).

For each $\phi\in X_{\frac{1}{2}}$ by the assumption (P2), we see that

$$\begin{aligned}<I_k(\phi),e_n> &= \int_0^1\left(I\left(x,\int_0^1\phi(y)dy\right)\right).\sqrt{2}sin(n\pi x)dx\\ &= \frac{1}{n\pi}\int_0^1\left(\frac{\partial}{\partial x}I\left(x,\int_0^1\phi(y)dy\right)\right).\sqrt{2}cos(n\pi x)dx\end{aligned}$$

Hence, I_k is a function from $X_{\frac{1}{2}}$ into $X_{\frac{1}{2}}$ By P(2) (ii) and the Holder inequality, we have

$$\begin{aligned}\|I_k(\phi_1)-I_k(\phi_2)\|_{\frac{1}{2}}^2 &= \int_0^1\left|\frac{\partial}{\partial x}I\left(x,\int_0^1\phi_1(y)dy\right)-\frac{\partial}{\partial x}I\left(x,\int_0^1\phi_2(y)dy\right)\right|^2dx\\ &\leq N_1^2\left|\int_0^1\phi_1(y)dy-\int_0^1\phi_2(y)dy\right|^2\\ &\leq N_1^2\|\phi_1-\phi_2\|_{\frac{1}{2}}^2\end{aligned}$$

for each $\phi_1,\phi_2\in X_{\frac{1}{2}}$. This implies that assumption (H4) holds. Thus system in the example is the abstract formulation of system (1.1).

We can take $q\frac{1}{2}$ and $f(t,\omega(t)=\frac{1}{t^{\frac{1}{3}}}sin\omega(t)$ Since for any $t\in I$, we have $\| f(t,\omega(t)\|=\|\frac{1}{t^{\frac{1}{3}}}sin\omega(t)\|\leq\frac{1}{3}$. So we choose $F(t)=\frac{1}{3}$, then assumptions (H1) and (H2) hold and the system is controllable.

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